THE FOCUSING NLS EQUATION ON THE HALF-LINE WITH PERIODIC BOUNDARY CONDITIONS: INSTABILITY OF THE DIRICHLET TO NEUMANN MAP

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Abstract. We consider the Dirichlet problem for the focusing NLS equation on the half-line, with given Schwartz initial data and boundary data $q(0,t)$ equal to an exponentially decaying perturbation $u(t)$ of the periodic boundary data $ae^{2i\omega t+i\epsilon}$ at $x=0$. It is known from PDE theory that this is a well-posed problem (for fixed initial data and fixed u). On the other hand, the associated inverse scattering transform formalism involves the Neumann boundary value for $x=0$. Thus the implementation of this formalism requires the understanding of the "Dirichlet-to-Neumann" map which characterises the associated Neumann boundary value.

We consider this map in an indirect way: we postulate a certain Riemann-Hilbert problem and then prove that the solution of the initial-boundary value problem for the focusing NLS constructed through this Riemann-Hilbert problem satisfies all the required properties: the data $q(x,0)$ are Schwartz and $q(0,t)-ae^{2i\omega t+i\epsilon}$ is exponentially decaying.

More specifically, we focus on the case $-3a^2<\omega<a^2$. By considering a large class of appropriate scattering data for the t-problem, we provide solutions of the above Dirichlet problem such that the data $q_x(0,t)$ is given by an exponentially decaying perturbation of the function $2iabe^{2i\omega t+i\epsilon}$, where $\omega=a^2-2b^2,\ b>0$.

On the other hand for periodic data exactly equal to $ae^{2i\omega t+i\epsilon}$ at $x=0$, in the case $\frac{a^2}{2}\leq\omega$, the data $q_x(0,t)$ is given (exactly) by the different function $2a\hat{b}e^{2i\omega t+i\epsilon}$, where $\omega=\frac{a^2}{2}+2\hat{b}^2,\ \hat{b}>0$. In other words, the Dirichlet to Neumann map is unstable in the sense that exponentially decaying perturbations of the boundary data $q(0,t)$ can lead to completely different data $q_x(0,t)$.

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1. Introduction

We are interested in classical solutions of the following initial-boundary value problem

$$
\begin{aligned}
iq_t(x,t) + q_{xx}(x,t) + 2|q(x,t)|^2 q(x,t) = 0,\ x>0,\ t>0,\\
q(x,0) = q_0(x),\ 0<x<\infty,\\
q(0,t) = g_0(t),\ 0<t<\infty,
\end{aligned}
\tag{1.1}
$$

where the function $q_0(x)$ belongs to the Schwartz class and $g_0(t) = ae^{2i\omega t+i\epsilon} + u(t)$, where $a>0, \omega, \epsilon$ are real, $u(t)$ decays exponentially as $t\to\infty$, and the compatibility condition $q_0(0) = g_0(0)$ is satisfied. We will assume here that $-3a^2 < \omega < a^2$.

It is known [2] that there exists a unique classical solution of this problem (for fixed u, q_0). On the other hand, the inverse scattering transform formalism developed in ([6], [7], [1]), in addition to $q_0(x)$ and $g_0(t)$ also requires the function $g_1(t) = q_x(0,t)$ for $0<t<\infty$. The general methodology of [5] is applied to the problem (1.1) in [1], where it is *assumed* that the *unknown* function g_1 is the sum of $2iabe^{2i\omega t+i\epsilon}$ (where $\omega = a^2 - 2b^2,\ b>0$) and a Schwartz function. (In fact, [1] consider only the case $u=0$, but their results go through even if u is exponentially decaying, or, say, Schwartz.)

It is known [8] that this assumption is not always true. Here is a counterexample for $a^2/2 \le \omega < a^2$, when $u=0$: consider the exact one breather solution

$$
q(x,t) = 2\eta e^{i\epsilon}\frac{e^{4i\eta^2 t}}{cosh 2\eta(x-x_0)}.
\tag{1.2}
$$

Clearly $q(x,0)$ is Schwartz and $q(0,t) = ae^{2i\omega t+i\epsilon}$ where $a = \frac{2\eta}{cosh(2\eta x_0)}$ and $\omega = 2\eta^2$. So $\omega \ge a^2/2$ but $q_x(0,t) = 2a\hat{b}e^{2i\omega t+i\epsilon}$ where $\hat{b} = \eta\ tanh(2\eta x_0)$ and $\hat{b}^2 = \omega/2 - a^2/4$.

The aim of this paper is to prove that this assumption *is* correct for at least some exponentially decaying u. Since the above counterexample shows that it is not true for all such u we deduce that the Dirichlet to Neumann map for the above initial boundary value problem (the map that takes $q(0,t)$ to $q_x(0,t)$) is highly unstable.

2. A Riemann-Hilbert problem

The focusing NLS equation admits the Lax pair

$$\mu_x + ik[\sigma_3, \mu] = Q(x,t)\mu, \tag{2.1a}$$

$$\mu_t + 2ik^2[\sigma_3, \mu] = \tilde{Q}(x,t,k)\mu, \tag{2.1b}$$

where $\sigma_3 = \mathrm{diag}(1,-1)$,
(2.2)

$$Q(x,t) = \begin{bmatrix} 0 & q(x,t) \\ -\bar{q}(x,t) & 0 \end{bmatrix}, \qquad \tilde{Q}(x,t,k) = 2kQ - iQ_x\sigma_3 + i|q|^2\sigma_3.$$

A novel method for analysing initial boundary value problems for integrable nonlinear PDEs was introduced in [5]. This method, which is based on the *simultaneous* spectral analysis of both the x-problem and the t-problem in the Lax pair, was rigorously implemented to the NLS on the half-line with Schwartz initial and boundary conditions in [7]. In the problem (1.1) the initial data are of Schwartz class, thus the scattering and inverse scattering of the x-problem is classical and goes back to the original investigations of Gelfand, Levitan and Marchenko (see [7]). On the other hand, the boundary values at $x = 0$ are perturbations of finite-zone functions, thus the spectral analysis of the t-problem involves aspects of the finite-zone theory. In this paper we will consider the simplest possible case of zero-zone data.

The zero-zone solution of NLS, namely $q(x,t) = q_p(x,t) = ae^{2ibx+2i\omega t+i\epsilon}$ gives rise to the Dirichlet data $ae^{2i\omega t+i\epsilon}$ and also yields $q_x(0,t) = 2iabe^{2i\omega t+i\epsilon}$.

Now, let b be defined by $\omega = a^2 - 2b^2,\ b > 0$. We will assume here that $a^2 - \omega > 0$ and $b^2 < 2a^2$. Let $\Omega(k)$ be the function defined as

$$\Omega(k) = 2(k-b)X(k),\ \ X(k) = \sqrt{(k+b)^2 + a^2}. \tag{2.3}$$

Following [1] we consider the two-sheeted Riemann surface X defined by the function $\Omega(k)$. Our Riemann-Hilbert problem will be defined on X. We also consider the oriented contour Σ defined by $Im\Omega(k) = 0$, see Figure 1. (This is Figure 9 of [1] with some contours reoriented.)

One easily sees that the curve Σ consists of two copies of the real line and an analytic arc $\Gamma \cup \bar{\Gamma}$ connecting the two branch points $E = -b+ia, \bar{E} = b-ia$ and the two infinities ∞_1 and ∞_2 (on the two sheets of X).

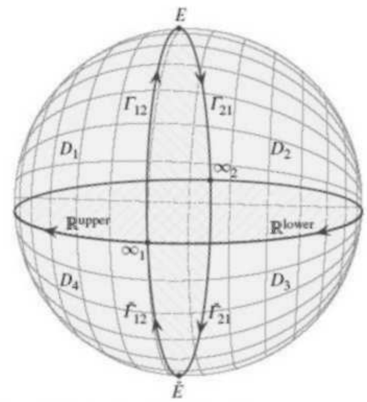

FIGURE 1. The two-sheeted Riemann surface X.

Σ defines a partition of the sphere X into D_1, D_2, D_3, D_4, where

$$
\begin{aligned}
D_1 &= \{Imk > 0, Im\Omega(k) > 0\}, \\
D_2 &= \{Imk > 0, Im\Omega(k) < 0\}, \\
D_3 &= \{Imk < 0, Im\Omega(k) > 0\}, \\
D_4 &= \{Imk < 0, Im\Omega(k) < 0\}.
\end{aligned}
\tag{2.4}
$$

Next, define the following matrices

$$
E(k) = \begin{pmatrix} (\frac{k+b+X(k)}{2X(k)})^{1/2} & ie^{i\epsilon}(\frac{X(k)-k-b}{2X(k)})^{1/2} \\ ie^{-i\epsilon}(\frac{X(k)-k-b}{2X(k)})^{1/2} & (\frac{k+b+X(k)}{2X(k)})^{1/2} \end{pmatrix}, \tag{2.5}
$$

$$
\begin{aligned}
&H(t,k) = exp(i\omega\sigma_3 t)E(k)exp(-i\omega\sigma_3 t), \\
&\Psi(t,k) = H(t,k)exp(i[\omega - \Omega(k)]\sigma_3 t).
\end{aligned}
$$

Let the functions $a(k)$ and $b(k)$ be the (classical) scattering data for the function $q_0(x)$ defined in [7]. All we need to know here is that $a(k)$ is smooth for k real and can be analytically extended in the upper half-plane, with $a(k) = 1 + O(1/k)$ as $k \to \infty$. Similarly, $b(k)$ is a Schwartz function for k real which can be extended to the upper half-plane such that $b(k) = O(1/k)$ as $k \to \infty$. Furthermore, $|a^2| + |b^2| = 1$ for k real and a can have at most a finite number of simple zeros in the complex plane, say $k_1, k_2, ..., k_n$, with $Im(k_j) > 0, j = 1, ..., n$.

Let the functions A, B be functions satisfying the following conditions:

(i) The functions $A(k), B(k)$ are analytic in $D_1 \cup D_3$, bounded in $\bar{D}_1 \cup \bar{D}_3$ and satisfy the following asymptotics $A(k) = 1 + O(1/k),\ B(k) = O(1/k)$ as $k \to \infty$.

(ii) $b(k)A(k) - a(k)B(k) = 0$ in D_1. This is the so-called global relation.

(iii) $A(k)\bar{A}(\bar{k}) + B(k)\bar{B}(\bar{k}) = 1,\ A(k) \neq 0,\ k \in \Sigma$.

We will now define a Riemann-Hilbert problem in X, with jump data given in terms of a, b, A, B, following [1].

We define the matrices

$$s(k) = \begin{pmatrix} \bar{a}(\bar{k}) & b(k) \\ -\bar{b}(\bar{k}) & a(k) \end{pmatrix}, \tag{2.6}$$

$$S(k) = \begin{pmatrix} \bar{A}(\bar{k}) & B(k) \\ -\bar{B}(\bar{k}) & A(k) \end{pmatrix} \tag{2.7}$$

and $G(k) = s^{-1}(k)S(k)$. Let

$$\rho(k) = \frac{G_{21}(k)}{G_{11}(k)},$$
$$r(k) = \frac{\bar{b}(k)}{a(k)},$$
$$c(k) = \rho(k) - r(k).$$

Consider now the following Riemann-Hilbert problem with the jump contour Σ:

$$\begin{aligned} M_-(x,t,k) &= M_+(x,t,k)J(x,t,k),\ k \in \Sigma, \\ lim_{k\to\infty_1} M(x,t,k) &= I, \end{aligned} \tag{2.8}$$

where

(2.9)

$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{r}(k)e^{-2i(kx+(\Omega(k)-\omega)t)} \\ r(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1+|r(k))|^2 \end{pmatrix},\ k \in \mathbb{R}^{upper},$$

$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{\rho}(k)e^{-2i(kx+(\Omega(k)-\omega)t)} \\ \rho(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1+|\rho(k))|^2 \end{pmatrix},\ k \in \mathbb{R}^{lower},$$

$$J(x,t,k) = \begin{pmatrix} 1 & 0 \\ c^+(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1 \end{pmatrix},\ k \in \Gamma_{12},$$

$$J(x,t,k) = \begin{pmatrix} 1 & 0 \\ -c^-(k)e^{2i(kx+(\Omega(k)-\omega)t)} & 1 \end{pmatrix},\ k \in \Gamma_{21},$$

$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{c}^+(\bar{k})e^{-2i(kx+(\Omega(k)-\omega)t)} \\ 0 & 1 \end{pmatrix},\ k \in \bar{\Gamma}_{12},$$

$$J(x,t,k) = \begin{pmatrix} 1 & -\bar{c}^-(\bar{k})e^{-2i(kx+(\Omega(k)-\omega)t)} \\ 0 & 1 \end{pmatrix},\ k \in \bar{\Gamma}_{21}.$$

Here c_+ and c_- are boundary values of the function c which is analytic in D_2.

Furthermore the following pole conditions are satisfied.

(2.10)
$$\begin{aligned} res_{k=k_j}[M(x,t,k)]_1 &= im_j^1 e^{2i(kx+(\Omega(k_j)-\omega)t)}[M(x,t,k_j)]_2,\ k_j \in D_1, \\ res_{k=z_j}[M(x,t,k)]_1 &= im_j^2 e^{2i(kx+(\Omega(z_j)-\omega)t)}[M(x,t,z_j)]_2,\ z_j \in D_2, \\ res_{k=\bar{z}_j}[M(x,t,k)]_2 &= -i\bar{m}_j^2 e^{-2i(\bar{k}x+(\Omega(\bar{z}_j)-\omega)t)}[M(x,t,\bar{z}_j)]_1,\ \bar{z}_j \in D_3, \\ res_{k=\bar{k}_j}[M(x,t,k)]_2 &= -i\bar{m}_j^1 e^{2i(\bar{k}x+(\Omega(\bar{k}_j)-\omega)t)}[M(x,t,\bar{k}_j)]_1,\ \bar{k}_j \in D_4, \end{aligned}$$

where

$$\begin{aligned} m_j^1 &= (ib(k_j)\frac{da}{dk}(k_j))^{-1},\ m_j^2 = -res_{k=z_j}c(k), \\ \bar{m}_j^1 &= (i\bar{b}(\bar{k}_j)\frac{d\bar{a}}{dk}(\bar{k}_j))^{-1},\ \bar{m}_j^2 = -res_{k=\bar{z}_j}\bar{c}(\bar{k}). \end{aligned} \tag{2.11}$$

Theorem 2.1. *The above Riemann-Hilbert problem admits a unique solution.*

The theorem follows immediately from the so-called vanishing lemma extended to the surface X [9] by employing the symmetries of the jump J. Although the vanishing lemma applies to holomorphic Riemann-Hilbert problems, the above meromorphic Riemann-Hilbert problem can be easily transformed to a holomorphic Riemann-Hilbert problem as in [3] by adding small loops around the poles and changing variables inside the loops (see also [6], [7]).

3. Asymptotic analysis of the Riemann-Hilbert problem

The analysis in section 3.3 of [1] shows that the Riemann-Hilbert problem above gives rise to a solution of the focusing NLS in the first quadrant. Furthermore the initial data $q(x,0)$ are equal to q_0 because of the definition of a, b. What is not a priori clear is that $q(0,t) = g_0(t)+u(t)$ and $q_x(0,t) = 2iabe^{2i\omega t+i\epsilon}+v(t)$, where u, v are exponentially decaying at infinity.

This is the main result of this paper.

Theorem 3.1. *Define $q(x,t) = 2ilim_{k\to\infty_1} kM_{12}(x,t,k)$ where M_{12} is the (12) entry of the solution of the above Riemann-Hilbert problem. Then $q(x,t)$ solves the focusing NLS equation in the first quadrant, with $q(x,0) = q_0(x),\ q(0,t) = g_0(t)+u(t),\ q_x(0,t) = 2iabe^{2i\omega t+i\epsilon}+v(t)$ where $u(t), v(t)$ are exponentially decaying at infinity.*

PROOF: Follows from the asymptotic analysis of the Riemann-Hilbert problem (for data a, b, A, B), as $t \to \infty$. From section 3.3 of [1] we have that the Riemann-Hilbert problem above reduces to the following Riemann-Hilbert problem when $x = 0$:

$$
\begin{aligned}
M_-^{(t)}(t,k) &= M_+^{(t)}(t,k)J^{(t)}(t,k),\ k\in\Sigma,\\
& lim_{k\to\infty_1}M^{(t)}(t,k)=I,
\end{aligned} \tag{3.1}
$$

where

$$
J^{(t)}(t,k)=\begin{pmatrix} 1 & \frac{B(k)}{A(k)}e^{-2i(\Omega-\omega)t} \\ \frac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}e^{2i(\Omega-\omega)t} & \frac{1}{A(k)\bar{A}(\bar{k})}\end{pmatrix},\ k\in\Sigma, \tag{3.2}
$$

where the superscript $+$ denotes the limit from the $+$side of the contour and the superscript $-$ denotes the limit from the $-$side of the contour.

The following asymptotic analysis will show that as $t\to\infty$, we recover the pure zero-zone solution.

Theorem 3.2. *Up to an exponentially small error, the Riemann-Hilbert problem for $M^{(t)}$ is asymptotically (as $t\to\infty$) equivalent to the trivial Riemann-Hilbert problem which has no jump. By this we mean that $lim_{k\to\infty}(k^n M_{12}^{(t)})$ is exponentially small for $n=1,2,\ldots$.*

Proof. Note the factorization of $J^{(t)}$ on $\mathbb{R}^{upper}\cup\bar{\Gamma}$:

$$
\begin{aligned}
& J^{(t)}(t,k)=J^{up}J^{lo},\\
& where\ J^{up}=\begin{pmatrix} 1/D_- & B(k)\bar{A}(\bar{k})D_-e^{-2i(\Omega-\omega)t} \\ 0 & D_-\end{pmatrix},\\
& and\ J^{lo}=\begin{pmatrix} D_+ & 0 \\ A(k)\bar{B}(\bar{k})(D_+)^{-1}e^{2i(\Omega-\omega)t} & 1/D_+\end{pmatrix},
\end{aligned} \tag{3.3}
$$

and where D solves the scalar problem

$$
D_+=D_-A(k)\bar{A}(\bar{k}),\ k\in\mathbb{R}^{upper}\cup\bar{\Gamma}
$$

and satisfies $lim_{k\to\infty_1}D(k)=1$. This factorization follows from the identity $A(k)\bar{A}(\bar{k})+B(k)\bar{B}(\bar{k})=1$ for $k\in\Sigma$.

Similarly, note the factorization of $J^{(t)}$ on $\mathbb{R}^{lower}\cup\Gamma$:

$$
\begin{aligned}
& J^{(t)}(t,k)=G^{lo}G^{up},\\
& where\ G^{lo}=\begin{pmatrix} 1 & 0 \\ \frac{\bar{B}(\bar{k})}{\bar{A}(\bar{k})}e^{2i(\Omega-\omega)t} & 1\end{pmatrix},\\
& G^{up}=\begin{pmatrix} 1 & \frac{B(k)}{A(k)}e^{-2i(\Omega-\omega)t} \\ 0 & 1\end{pmatrix}.
\end{aligned} \tag{3.4}
$$

For the asymptotic analysis we must deform our Riemann-Hilbert problem in small lenses with boundaries consisting of the different components of $\mathbb{R}\cup\Gamma$ and slight deformations of these components.

For example we consider the oriented contours $C^{1,up}$ and $C^{1,lo}$ from ∞_1 to ∞_2 on the upper sheet of the Riemann surface slightly deforming the real line, with $C^{1,up}$ lying in D_1 and $C^{1,lo}$ lying in D_4, and denote the corresponding lenses $D_{1,up}$ and $D_{1,lo}$ in a way that $\partial D_{1,up} = C^{1,up} \cup \mathbb{R}^{upper}$ and $\partial D_{2,up} = C^{2,up} \cup \mathbb{R}^{upper}$. We construct similar lenses around $\Gamma, \bar{\Gamma}, \mathbb{R}^{lower}$.

We define O as follows:

$$O(t,k) = M^{(t)}(t,k) J^{up}, \ \ k \in D_{1,lo},$$
$$O(t,k) = M^{(t)}(t,k) (J^{lo})^{-1}, \ \ k \in D_{1,up}. \tag{3.5}$$

Similarly for the other lenses. Note that O is piecewise analytic off Σ only if A, B are analytic in the appropriate lenses. This is not assumed to be generallly true, but A, B can always be approximated by analytic functions in a way that the overall error due to the substitution of A, B by their analytic approximations is exponentially small as $t \to \infty$ (see [4]).

We now observe that the off-diagonal entries of the jump matrix for O are $uniformly$ exponentially small. On the other hand, the diagonal entries are uniformly bounded. So, according to standard asymptotic analysis of Riemann-Hilbert factorization problems [4], it follows that, up to an exponentially small error, O is given by the solution of a problem with diagonal jump, which in turn reduces to the scalar problem for D. The off-diagonal entries of $M^{(t)}$ thus have to be exponentially small in t, to all orders in k. $\square$

The limiting Riemann-Hilbert problem is trivial and corresponds to the purely zero-zone solution of NLS. Using the formulae $q(0,t) = lim_{k\to\infty}[2ikM_{12}(0,t)]$ and $q_x(0,t) = lim_{k\to\infty}[4k^2 M_{12}(0,t) + 2iq(0,t)kM_{22}(0,t)]$ we see that $u(t), v(t)$ are actually exponentially small. Theorem 3.1 is thus proved. In fact, using similar formulae for $\frac{\partial^j}{\partial t^j} v(t), \ i = 1,2,3,....$ in terms of $M^{(t)}$ it is possible to show that u, v are Schwartz functions.

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